Absorbing Phase Transition in a Four State Predator Prey Model in One Dimension

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Abstract. The model of competition between densities of two different species, called predator and prey, is studied on a one dimensional periodic lattice, where each site can be in one of the four states say, empty, or occupied by a single predator, or occupied by a single prey, or by both. Along with the pairwise death of predators and growth of preys, we introduce an interaction where the predators can eat one of the neighboring prey and reproduce a new predator there instantly. The model shows a non-equilibrium phase transition into a unusual absorbing state where predators are absent and the lattice is fully occupied by preys. The critical exponents of the system are found to be different from that of the Directed Percolation universality class and they are robust against addition of explicit diffusion.

Absorbing configurations do not have any outgoing rates [1]. Once reached there, the system can not escape from these configurations. Presence of absorbing configurations in a phase space raises a possibility that the concerned system may undergo a non-equilibrium phase transition into absorbing states. The critical behavior of these absorbing state phase transitions (APT)s [2] depends on the conservation in dynamics and the symmetry between absorbing states. It has been conjectured [3] that in absence of any special symmetry the APT belong to the directed percolation (DP) universality class as long as the system has a single absorbing state. Additional symmetries, like particle-hole symmetry [4], conservation of parity [5], and symmetry between different absorbing states [6] lead to different universalities. Spreading process with spatially quenched randomness [7] or with long-term memory [8] are known to destroy the critical behavior completely, whereas the long-range interaction leads to continuous variation [9] of critical exponents. Presence of infinitely many absorbing states may [10] or may not [11] belong to DP universality class. Again a different critical behavior is observed when the activity field does not have any special symmetry, but it is coupled to a conserved density [12]. Recent studies have indicated that DP-critical behavior is possible, even in presence of an additional conserved field [13]. It is not quite clear, what microscopic ingredients can make an APT belong to the DP class.

The models of directed percolation has been extended to more than one species [14]. Along with the simple DP behavior, a line of first order transition [15] has been observed in $1+1$ dimension when two species compete for survival. Janssen [16] studied coupled DP processes with bilinear and bidirectional interspecies couplings in the framework of bosonic field theory, where no other critical phenomena were found other than the DP. Hierarchy of unidirectionally coupled DP processes with many species show multicritical behavior [17]. Coupled percolation processes have been also studied [18], where the absorbing phase become unstable with respect to an arbitrarily small branching rate even in one dimension.

Predator-prey cellular automaton models [19] in two dimension show DP universality class. Coupled directed percolation (DP) processes with more than two species of particles (in one dimension) with different kind of interspecies coupling have shown DP-type [20] transitions. Lotka-Volterra like models in one dimension always show coexistence [21], either in form of well mixed states or as irregular bursts of the predator and prey population. A four state predator prey model [22] in one dimension with a restriction that a site can have at best one particle of each kind, shows an APT to an absorbing (extinct) state which belongs to DP-class.

In this article we study a model of two species, say A (prey) and B (predator), on a $(1+1)$-dimensional lattice. Each lattice site is either vacant $\emptyset$ or occupied by at best one particle of each kind. The preys grow independently as $A\emptyset \rightarrow AA$ and the predators die as $BB \rightarrow \emptyset\emptyset$, whereas they interact through a process $BA \rightarrow BB$, where birth of a new predator occurs instantly along with the death of the prey. The system show a line of continuous absorbing state transition different from DP as the rates of these processes are tuned. Unlike other multispecies models, in the absorbing state
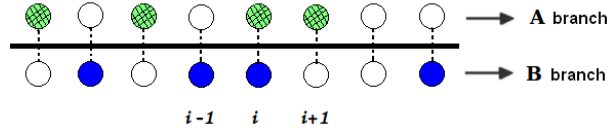
Figure 1. Schematic diagram of the 4-state predator-prey (4SPP) model

both preys and predators do not separately extinct, rather predators extinct and preys proliferate to fill up the whole prey branch. The model is defined on a one dimensional periodic lattice with lattice sites labelled by $i = 1, 2 \ldots L$. Each site i can be either vacant or occupied by a single particle A (prey), or occupied by a single particle B (predator) or by both particles (co-existing A and B), thus the model can be treated as 4-state predator-prey (4SPP) model. More than one particle of any kind is not allowed. These hardcore restrictions on individual particles, where co-existence is allowed, can be realized alternatively by considering two separate branches, one for A and the other for B particles. Particles living in one branch can not move to the other branch, as schematically shown in the Fig. 1. Correspondingly, each site i is associated with four states; $s_A^i = 0, 1$ and $s_B^i = 0, 1$, where 1 (0) denotes the presence (absence) of a particle at site i.

On a periodic lattice, these particles interact following a random sequential dynamics given below. The prey (A) can grow on their own branch with rate
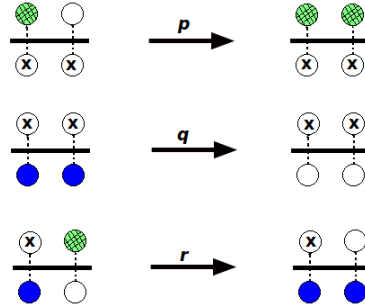
p independent of the predators (B). Again, two neighboring predators (B) die simultaneously with rate q independent of A due to their own crowding or overpopulation. Here X in A-branch (B-branch) corresponds to an arbitrary state of A (B) particles i.e. presence or absence. These two species interact with rate r as follows; when a predator B at site i meets a prey A as the right neighbor $i+1$, with site $i+1$ is not already occupied by a predator B, then the predator B eats the prey A and simultaneously reproduces another predator B. Note, that the dynamics is spatially asymmetric as both species grow only in one direction (here, right), and it does not include explicit diffusion of species. Effect on addition of symmetry and diffusion are briefly discussed towards the end.

Let the steady state density of A and B particles be ρ_A^s and ρ_B^s respectively. Clearly in absence of predators (ρ_B^s) the prey density $\rho_A^s = 1$ as the preys do not have an

independent death process. Again, note that the dynamics do not allow A particles to be eaten when they co-exist with B on the same site. This indicates that these As can only die after their coexisting Bs die, which can occur with rate q if there is a B particle present or created at the immediate left neighbor. Thus, for reasonably small death rate of predators, ρ_A^s is expected to have a small value ($\rho_A^s < 1$) when $\rho_B^s \simeq 0$ and then it increases along with ρ_B^s; so, the prey density ρ_A^s can never vanish. The predator density ρ_B^s can, however, become zero by repetitive death process. The isolated Bs wait until the prey invades their neighboring site and then they subsequently eat and reproduce with rate r and die with rate q. So, along with the *coexisting phase* ‡ where both ρ_A^s and ρ_B^s are non-zero, we have another phase where $\rho_A^s = 1$ and $\rho_B^s = 0$. Clearly, the later phase ($\rho_A^s = 1, \rho_B^s = 0$) is absorbing as once all the predators die, even then the single surviving prey can lead to proliferation of the prey population in the whole lattice. Of course, as argued earlier, ρ_A^s can not vanish, and the other possible absorbing state ($\rho_A^s = 0, \rho_B^s = 0$) which requires simultaneous death of all predators and preys, is not dynamically accessible. Thus the 4SPP model can undergo an absorbing state phase transition by tuning the different rates with ρ_B^s as the order parameter. Our aim here is to study this critical behavior in details.

We have used the standard Monte Carlo methods to study the critical behavior of this model. From an initial arbitrary configuration, where each branch A and B are filled by arbitrary number of respective particles, the system is allowed to evolve according to the random sequential update following the Monte Carlo dynamics of 4SPP model. We have studied the system with different values of the rates p, q and r with system size $L = 10^3$. For illustration, we have fixed two of the reaction rates, say $q = 0.02$, $r = 0.9$, and vary p as the the control parameter in the following simulations.

For $p < p_c$ the average B density ρ_B^s decreases continuously until the system reaches a state with no predators ($\rho_B^s = 0$) and ultimately the whole prey lattice branch is filled with preys. Once reached in this absorbing state ($\rho_A^s = 1, \rho_B^s = 0$), the system remains there forever. While, for $p > p_c$ the average density of B particles ρ_B^s saturates to a nonzero value along with ρ_A^s which also takes a value smaller than unity. In Fig. 2(a) we have plotted ρ_A^s and ρ_B^s as a function of p for a system size $L = 10^3$. As expected, the density of the preys ρ_A^s never vanish and they proliferates in the whole lattice (with $\rho_A^s = 1$) in the absorbing state where predators are absent. Thus the system undergoes an absorbing state phase transition as the birth rate of preys p crosses a critical threshold $p_c = 0.148(4)$.

For $p > p_c$, the order parameter ρ_B^s shows power law behavior with the distance from criticality,

$$\rho_B^s \sim (p - p_c)^\beta, \tag{1}$$

when p approaches p_c. This Eq. (1) can be used to estimate p_c and β. As shown in Fig. 3(a), ρ_B^s versus $(p - p_c)$ is linear in log scale, for the correct choice of $p_c = 0.148(4)$;

‡ It has been predicted earlier [22] that, in absence of site restriction, the predator and the prey system in 1D always remain in the co-existing phase.
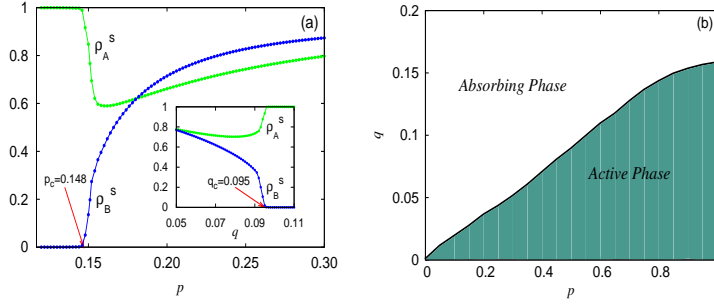
Figure 2. (a) Average steady state density ρ_A^s, ρ_B^s is plotted against the control parameter p for fixed value of $q = 0.02, r = 0.9$ and $L = 10^3$, critical point is indicated at $p_c = 0.148(4)$ showing absorbing to active phase transition. Both densities are again plotted against q keeping $p = 0.55$, $r = 0.9$ fixed; critical point is $q_c = 0.095(6)$ showing active to absorbing phase transition (see inset). (b) Phase diagram in the (p, q) plane for $r = 0.9$, showing active and absorbing phases.

the corresponding slope $\beta = 0.367(7)$ gives the estimated value of the order parameter exponent.

One can obtain few other critical exponents from the decay of the order parameter $\rho_B(t)$ from an initial configuration with large number of predators. Clearly, in the vicinity of critical point ρ_B is a function of time t and the temporal correlation length $\xi_\parallel$ which vary as $\xi_\parallel \sim |p - p_c|^{-\nu_\parallel}$. Again, after an initial decay $\rho_B(t) \sim t^{-\alpha}$ the predator density in the active phase approaches the steady state value ρ_B^s in the $t \to \infty$ limit. So ρ_B must scale as,

$$\rho_B(t, p) = t^{-\alpha} \mathcal{F}(t|p - p_c|^{\nu_\parallel}). \tag{2}$$

Thus, one expects that $\rho_B(t)$ for different values of p (shown in the Fig. 3(b)) collapsed into a single scaling function $\mathcal{F}$, when $\rho_B t^\alpha$ is plotted against $t|p - p_c|^{\nu_\parallel}$. The main figure
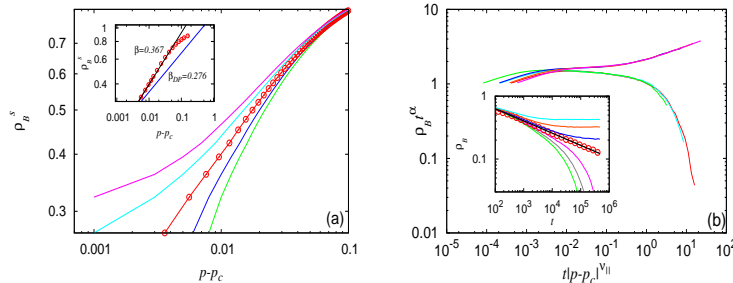

Figure 3. (a) Average predator density (ρ_B^s) is plotted with fixed value of $q = 0.02, r = 0.9$; with different choices of $p_c = 0.153, 0.151, 0.148, 0.146, 0.144$ shown from top to bottom. The correct choice of $p_c = 0.148(4)$ and corresponding slope in logarithmic plot gives critical exponent $\beta = 0.367(7)$. (b) Time evolution of ρ_B with $q = 0.02, r = 0.9$. Below the critical point for $p = 0.142, 0.144, 0.146$ density ρ_B eventually extincts and above the critical point $p = 0.150, 0.154, 0.160$ density ρ_B saturates. At the critical point $p = 0.148(4)$, $\rho_B(t) \sim t^{-\alpha}$ gives the critical exponent $\alpha = 0.194(4)$ and the data collapse gives $\nu_\parallel = 1.8(1)$.

here shows the data collapse when we choose $\alpha = 0.194(4)$, and $\nu_{\shortparallel} = 1.8(1)$. Since at the critical point $\rho_B(t, p_c) = t^{-\alpha}\mathcal{F}(0)$, one can obtain both p_c and α directly from the log scale plot of ρ_B versus t which is linear (as shown in the inset of Fig. 3(b)). The resulting p_c and α are consistent with those obtained from the data collapse. Again, in the upper critical regime, ρ_B^s vanishes as $|p - p_c|^\beta$, in the $t \to \infty$ limit. This can happen only when the off-critical scaling function $\mathcal{F}(x) \sim x^{\beta/\nu_{\shortparallel}}$; thus

$$\alpha = \beta/\nu_{\shortparallel}$$

Since all three exponents β, α and $\nu_{\shortparallel}$ are calculated independently, one can check if the above scaling relation holds. In this case it holds to a great accuracy for the values of β, α and $\nu_{\shortparallel}$ calculated here.

Now we turn our attention to the finite size scaling of ρ_B at the critical point. Again, the system of length L with a high density of predators $\rho_B(t, L)$ decays as $t^{-\alpha}$, indicating a scaling form

$$\rho_B(t, L) = t^{-\alpha}\mathcal{G}(t/L^z), \tag{3}$$

where z is the dynamic critical exponent. Thus, one expects ρ_B for different values of L to be collapsed to a single function when plotted against t/L^z. This is described in Fig. 4(a). The inset there shows variation of $\rho_B(t)$ for different system size $L = 50, 100, 200, 300, 400, 500$, which were made to collapse to a single function using $\alpha = 0.194(4)$ and $z = 1.52(0)$. From the scaling relation

$$z = \nu_{\shortparallel}/\nu_\perp,$$

one expects that $\nu_\perp = 1.19$. This can be verified from the modified scaling relation Eq. (3). Since $z\alpha = \beta/\nu_\perp$, we have

$$\rho_B(t, L) = L^{-\beta/\nu_\perp}\tilde{\mathcal{G}}(t/L^z), \tag{4}$$

where $\tilde{\mathcal{G}}(x) = x\mathcal{G}(x)$. In Fig. 4(b) we have plotted $\rho_B L^{\beta/\nu_\perp}$ as a function of t/L^z and found that the data for system size $L = 50, 100, 200, 300, 400$, and 500 could be
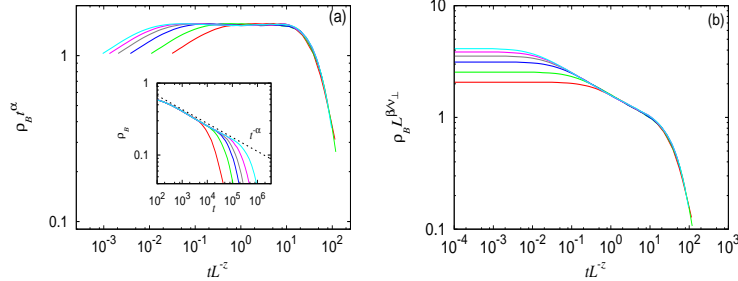

Figure 4. (a) Finite size scaling function $\rho_B t^\alpha$ plotted against the scaled variable tL^{-z} for $L = 50, 100, 200, 300, 400, 500$ (bottom to top in figures) with $p = 0.148(4)$, estimates $z = 1.52(0)$. (b) The same data could be collapsed according to Eq. (4) by choosing $\beta/\nu_\perp = 0.30$, which gives an estimate $\nu_\perp = 1.2(2)$.

collapsed into a single curve by choosing $\beta/\nu_\perp = 0.30$. This gives us $\nu_\perp = 1.2(2)$, which is consistent with the earlier estimation.

The critical exponents of the model are summarized in Table-1 along with the the critical exponents of directed percolation universality class. Clearly the exponents, particularly β and α, are very different from those of DP-class, indicating that the 4SPP model belongs to a different universality class. We have studied the model by varying other rates also.

	β	α	$\nu_\parallel$	z	$\nu_\perp$
DP	0.276	0.159	1.733	1.580	1.096
4SPP	0.367(7)	0.194(4)	1.8(1)	1.52(0)	1.2(2)

Table 1. Comparison of the critical exponents between the 4SPP model with the DP universality class.

For example, one can take q as the control parameter, keeping p and r fixed. The inset of Fig. 2(a) shows the variation of ρ_A^s and ρ_B^s as a function of q for fixed $p = 0.55$ and $r = 0.9$. Evidently, the order parameter ρ_B^s vanishes continuously as q crosses the threshold value $q_c = 0.095(6)$. The critical exponents obtained in this case was found to be consistent with Table-1. Thus, for any fixed values of r, one expects a line of criticality in the (p, q)-plane, which is shown in Fig. 2(b).

That, the critical behavior of the absorbing phase transition observed in the 4SPP model is different from DP, can be visualized from the growth of clusters. The space-time diagram, starting from an arbitrary initial configuration is shown in Fig. 5(a), where the occupancy of species A and B are represented separately in the upper and lower part respectively. The evolution of clusters are visibly different from that of $1+1$ dimensional directed percolation model. Since the prey species can grow independently, the space is always filled locally by preys where predators are absent. Again, regular *striped* structures appear in these figures as the species do not diffuse. A natural question would be whether diffusion can drive the system to have an absorbing state transition belongs to DP universality class or not.

In the following we introduce diffusion of both the species *explicitly* in this model. Along with the usual dynamics of the 4SPP model described earlier, both A and B particles are now allowed to move to the neighboring available vacant space in their respective branches. In this case, it turns out that the clusters evolve more like the DP model (see the space-time diagram in Fig. 5(b)). However, the detailed study of the absorbing phase transition reveals that the critical exponents are same as given in Table-1.

In presence of *explicit diffusion*, we choose to study the system with fixed rates $q = 0.2, r = 0.9$. Monte-carlo simulations show that the predator density ρ_B^s vanishes continuously as p is decreases below a critical threshold $p_c = 0.323(5)$. As described in Fig. 6(a), near the critical point, $\rho_B^s \sim (p - p_c)^\beta$ with $\beta = 0.370(9)$. Again, starting from
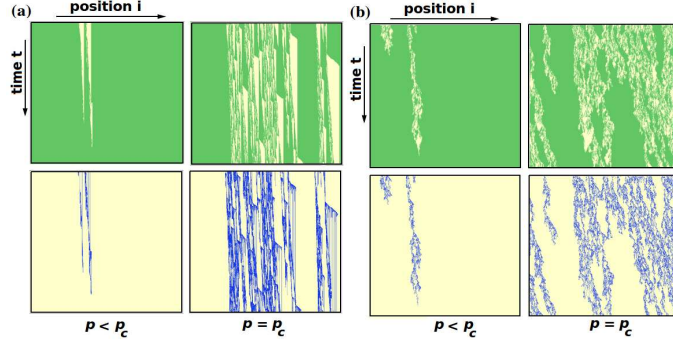
Figure 5. Time evolution of the 4SPP model for 10^3 time steps with prey A (green) and predator B (blue) with system size $L = 10^3$ for the regime $p < p_c$ and at $p = p_c$: (a) with asymmetric rules (no diffusion), (b) with asymmetric rules and *explicit diffusion*.

a large number of predators, the density $\rho_B(t)$ decays to its stationary value ρ_B^s which is nonzero only in the upper-critical region $p > p_c$. We find that $\rho_B(t)$ for different values of p could be merged to an unique scaling function which satisfy Eq. (2) by choosing $\alpha = 0.190(5)$ and $\nu_{\|} = 1.7(5)$. This data collapse is described in Fig. 6(b). The critical exponents β, α and $\nu_{\|}$ are more or less consistent with those listed in Table-1. It is not surprising that addition of explicit diffusion did not alter the universal behavior. In fact, though slow, effective diffusion of predators was already occurring in the 4SPP model through the rates q and r.
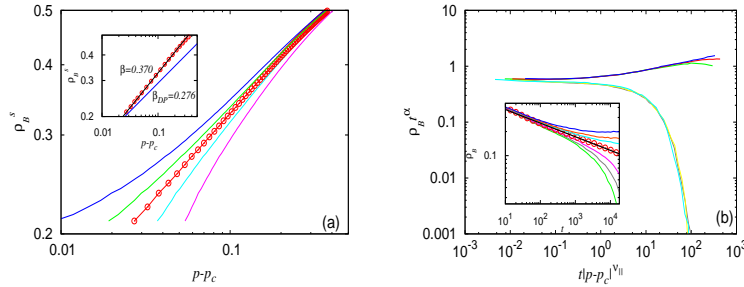

Figure 6. (a) Average density of predators (ρ_B^s) with $q = 0.2$, $r = 0.9$ with *explicit diffusion*: absorbing to active state phase transition at $p = 0.323(5)$ and the critical exponent $\beta = 0.370(9)$ (see inset). (b) Above the critical point for $p = 0.330, 0.335, 0.345$ the density ρ_B^s saturates and below the critical point for $p = 0.305, 0.310, 0.315$ density ρ_B^s eventually extincts. At the critical point $p = 0.323(5)$, $\rho_B(t) \sim t^{-\alpha}$ gives the critical exponent $\alpha = 0.190(5)$ and $\nu_{\|} = 1.7(5)$.

Some comments are in order here. The fact that the absorbing state phase transition in 4SPP model is different from that of DP can be seen from the dynamical mean-field analysis (ignoring spatial density correlations). The mean-field densities $\rho_A^i = \langle s_A^i \rangle$ and $\rho_B^i = \langle s_B^i \rangle$, in the continuum limit, evolve as,

$$\frac{\partial \rho_A}{\partial t} = p\rho_A(1 - \rho_A) - r\rho_A\rho_B - v_A\frac{\partial \rho_A}{\partial x} + D_A\frac{\partial^2 \rho_A}{\partial x^2} \qquad (5)$$

$$\frac{\partial \rho_B}{\partial t} = -2q\rho_B^2 + r\rho_A\rho_B - v_B\frac{\partial \rho_B}{\partial x} + D_B\frac{\partial^2 \rho_B}{\partial x^2}, \tag{6}$$

where $v_{A,B}$ denote the velocities of respective species which appear due to asymmetric dynamics, and $D_{A,B}$ are the coefficients of diffusion. The first term in Eq. (5) captures the growth of a prey restricted by hardcore interactions and the second term there corresponds to the interaction between two species. Again, the first term in Eq. (6) represents the simultaneous death of two predators. Evidently, these mean-field equations have two fixed points: the unstable one ($\rho_A^* = 0, \rho_B^* = 0$) and the stable one ($\rho_A^* = \frac{2pq}{2pq+r^2}, \rho_B^* = \frac{pr}{2pq+r^2}$). However, when $\rho_B \to 0$ faster than ρ_A, one can get another fixed point ($\rho_A^* = 1, \rho_B^* = 0$) from Eq. (5). This unusual absorbing state ($\rho_A^s = 1, \rho_B^s = 0$), as discussed earlier, raises a possibility that corresponding absorbing state phase transition can be different from the usual APT to ($\rho_A^* = 0, \rho_B^* = 0$). Clearly, for $\rho_A = 1$, the mean-field equation for the predator density ρ_B (from Eq. (6)) is identical to that of the DP. Thus, in higher dimension (larger than the critical dimension) one expects that the 4SPP model results in the same mean-field critical exponents ($\beta^{MF} = 1 = \nu_\parallel^{MF}, z^{MF} = 2$) as that of DP. Another possible reason for the new universality class is the asymmetric dynamics, which generates density dependent velocity terms $v_{A,B}$. Note that the absorbing state phase transition in the asymmetric contact process [23] belongs to the DP-class, whereas asymmetric updating is a relevant perturbation to models with extremal dynamics [24]. Detailed study of the 4SPP model with symmetric dynamics will be reported elsewhere.

In summary, we introduce a two species (predator B and prey A) model in one dimension where each lattice site is either vacant or occupied by, a single predator, a single prey or both. More than one predator or prey are not allowed at any site. The preys are allowed to grow independent of the predators, whereas two predators, if present at the neighboring sites, die simultaneously. The species interact through a dynamics where the predator produces an offspring by eating a prey from its neighbor. When the predator density $\rho_B^s = 0$, even a single prey can invade the whole lattice by its independent birth process. Thus, the system has an unusual absorbing state ($\rho_A^s = 1, \rho_B^s = 0$); the other absorbing state ($\rho_A^s = 0, \rho_B^s = 0$) is not dynamically accessible. Using dynamical Monte-Carlo simulation we show that the system shows an absorbing state phase transition, as the birth rate of prey p is increased beyond a critical value p_c keeping death rate of predator q fixed. For a fixed r, the line of criticality (p_c as a function of q) is governed by a universality class different from the most generic one, namely directed percolation. This critical behavior is found to be robust against addition of explicit diffusion. Note that the dynamical rules of the model is different from other four species predator prey models studied earlier [22] in a way that the predators in 4SPP model can not eat the prey at the same site. We believe that, non-equilibrium phase transition to a unusual and unique absorbing state, may result in different universality class. Further study in this direction could clarify this issue.

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